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McKean–Vlasov Limit for Interacting Random Processes in Random Media

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We apply large-deviation theory to particle systems with a random mean-field interaction in the McKean–Vlasov limit. In particular, we describe large deviations and normal fluctuations around the McKean–Vlasov equation. Due to the randomness in the interaction, the McKean–Vlasov equation is a collection of coupled PDEs indexed by the state space of the single components in the medium. As a result, the study of its solution and of the finite-size fluctuation around this solution requires some new ingredient as compared to existing techniques for nonrandom interaction.

KEY WORDS: Interacting particle systems; random media; McKean–Vlasov equation; large deviations; central limit theorem.

INTRODUCTION

In this paper, we consider interacting diffusions and interacting spin-flip systems with a mean-field Hamiltonian that depends on a random medium. In the thermodynamic limit, the dynamics of a typical particle is described by a collection of *coupled* McKean–Vlasov equations indexed by a medium parameter. For finite but large systems there are fluctuations around the McKean–Vlasov limit, which are controlled by the random dynamics and by the random medium.

Our approach to the problem is to do a large-deviation analysis for the *double-layer empirical measure*

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x^i_{[0,T]}, \omega^i)} \tag{0.1}$$

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Here, N is the size of the system and

$$\begin{aligned} x^i_{[0,T]} &= \text{the path of the } i\text{th particle in the time interval } [0, T] \\ \omega^i &= \text{the } i\text{th component of the medium} \end{aligned} \tag{0.2}$$

Our main results are the following (see Sections 1–3):

1. We derive a large-deviation principle for L_N as $N \to \infty$, with an explicit representation for the corresponding rate function I.
2. The McKean–Vlasov limit is the associated law of large numbers, i.e., the McKean–Vlasov equation follows from result 1 by identifying the unique zero of I.
3. By a standard contraction argument we derive a large-deviation principle for the *double-layer empirical flow*

$$l_N = \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{(x_t^i, \omega^i)} \right)_{t \in [0,T]} \tag{0.3}$$

   as $N \to \infty$ and compute the corresponding rate function i.
4. The second-order fluctuations around the McKean–Vlasov limit are identified in the form of a central limit theorem, deduced from result 1 via a variational computation.

The goal of our paper is twofold:

a. For homogeneous systems, results as in 1–4 have been obtained, among others, by Dawson,[7] Kusuoka and Tamura,[14] Dawson and Gärtner,[8] Ben Arous and Brunaud,[1] and Feng.[11] (See also Comets and Eisele[6] for models with a so-called "local" mean-field interaction.) We show how to generalize the analysis in these papers to systems with a random medium interaction. The random medium leads to some new ingredients in the analysis.

b. We want to give an expository presentation of the large-deviation approach to this problem area.

The outline of the paper is as follows. In Section 1 we consider interacting diffusions and state our theorems for this class of models (Theorems 1–4). Section 2 and Appendices A and B are devoted to the proof of the results. In Section 3 we consider spin-flip systems and show how the results have to be modified (Theorems 5–8).

1. DIFFUSIONS

1.1. The Model

Let $H_N\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the N-particle random Hamiltonian given by

$$H_N(\mathbf{x}, \boldsymbol{\omega}) = \frac{1}{2N} \sum_{i,j=1}^{N} f(x^j - x^i; \omega^i, \omega^j) + \sum_{i=1}^{N} g(x^i; \omega^i) \tag{1.1}$$

where $\mathbf{x} = (x^i)_{i=1}^N$ is the *state* variable and $\boldsymbol{\omega} = (\omega^i)_{i=1}^N$ is the *medium* variable. The ω^i are assumed to be i.i.d. random variables with common law μ. For a fixed realization of $\boldsymbol{\omega}$, think of $\mathbf{x} \to H_N(\mathbf{x}; \boldsymbol{\omega})$ as a Hamiltonian in the components x^i with an inhomogeneous mean field interaction parametrized by the components ω^i. The functions f and g play the role of a pair potential, resp. external field, and are assumed to satisfy:

- f, f', f'', g, g', g'' exist, are bounded, and are jointly continuous in all variables (a prime denotes derivative w.r.t. the x variable).[3]

For given $\boldsymbol{\omega}$, let $\mathbf{x}_t = (x_t^i)_{i=1}^N$ be the system of N interacting diffusions evolving according to the Itô stochastic differential equations

$$dx_t^i = -\frac{\partial H_N}{\partial x^i}(\mathbf{x}_t, \boldsymbol{\omega})\, dt + d\xi_t^i \qquad (i = 1,\ldots, N;\ t \in [0, T]) \tag{1.2}$$

where $(\xi_t^i)_{i=1}^N$ are i.i.d. standard Brownian motions on $\mathbb{R}$. For every $\boldsymbol{\omega}$, (1.2) has a reversible equilibrium measure proportional to $\exp[-H_N(\mathbf{x}, \boldsymbol{\omega})]$. The initial condition $\mathbf{x}_0$ is assumed to have product distribution $\lambda^{\otimes N}$, with λ having a finite second moment. The time $T > 0$ is fixed but arbitrary. Because f', g' are globally Lipschitz, (1.2) has a unique (strong) solution with continuous trajectories (see Karatzas and Shreeve,[13] Theorem 2.9).

We shall write P_N^{ω} to denote the law of $\mathbf{x}_{[0,T]} = (\mathbf{x}_t)_{t \in [0,T]}$ given $\boldsymbol{\omega}$, and $W^{\otimes N}$ to denote the law of the solution of (1.2) when $H_N \equiv 0$ (i.e., W is the law of a standard Brownian motion starting with initial distribution λ).

The system in (1.2) will be our object of study. We shall identify its large-deviation and central limit behavior in the limit as $N \to \infty$. Our main results are formulated in Theorems 1–4 in Sections 1.2–1.5 below.

[3] The assumptions on f, g are stronger than what is actually needed for proving the results in this paper. However, they allow us to illustrate the use of large deviations without excessive technicalities. A few more restrictions will be imposed later, for the same reason. For the medium variables $\mathbb{R}$ could be replaced by any Polish space without change in the proofs. For the state variables $\mathbb{R}$ could be replaced by $\mathbb{R}^d$ $(d \geqslant 1)$ with only minor modifications in the proof of Theorem 3 in Section 2.3.

1.2. Empirical Measure and Large Deviations

Define the *double-layer empirical measure*

$$L_N(\mathbf{x}_{[0,T]}, \boldsymbol{\omega}) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x^i_{[0,T]}, \omega^i)} \tag{1.3}$$

This is a random variable taking values in $\mathcal{M}_1(C[0, T] \times \mathbb{R})$, the set of probability measures on $C[0, T] \times \mathbb{R}$ (where $C[0, T]$ is the path space, i.e., the continuous functions on $[0, T]$). In (1.3), the symbol δ_y denotes the point measure at y, so

$$L_N(A) = \frac{1}{N} \sum_{i=1}^{N} 1\{(x^i_{[0,T]}, \omega^i) \in A\}, \qquad A \subset C[0, T] \times \mathbb{R}$$

Lemma 1 below gives a representation for P^ω_N in terms of L_N.

Lemma 1. For given $\boldsymbol{\omega}$

$$\frac{dP^\omega_N}{dW^{\otimes N}}(\mathbf{x}_{[0,T]}) = \exp[NF(L_N(\mathbf{x}_{[0,T]}, \boldsymbol{\omega}))] \tag{1.4}$$

where, for $Q \in \mathcal{M}_1(C[0, T] \times \mathbb{R})$,

$$\begin{aligned} F(Q) = \int & Q(dx_{[0,T]}, d\omega) \\ & \times \Bigg\{ -\tfrac{1}{2} \int_0^T dt \Bigg[\left(\int Q(dy_{[0,T]}, d\pi)\, \hat{f}'(y_t - x_t; \omega, \pi) + g'(x_t; \omega) \right)^2 \\ & - \int Q(dy_{[0,T]}, d\pi)\, \hat{f}''(y_t - x_t; \omega, \pi) + g''(x_t; \omega) \Bigg] \\ & - \tfrac{1}{2} \int Q(dy_{[0,T]}, d\pi)\, [f(y_T - x_T; \omega, \pi) - f(y_0 - x_0; \omega, \pi)] \\ & - [g(x_T; \omega) - g(x_0; \omega)] \Bigg\} \end{aligned} \tag{1.5}$$

with $\hat{f}$ given by

$$\hat{f}(x; \omega, \pi) = \tfrac{1}{2}[f(x; \omega, \pi) + f(-x; \pi, \omega)] \tag{1.6}$$

The proof of Lemma 1 will be given in Section 2.1. Note that $Q \to F(Q)$ is nonlinear and contains repeated integrals over the measure Q. A simpler representation for $F(Q)$ will be given in Lemma 2 below.

The representation in (1.4) is the key to the following large-deviation principle (LDP), from which we shall deduce various features of the asymptotic behavior of L_N as $N \to \infty$. Define

$$P_N(\cdot) = \int \mu^{\otimes N}(d\boldsymbol{\omega})\, P_N^{\boldsymbol{\omega}}(L_N \in \cdot) \tag{1.7}$$

which is the law of L_N under the joint distribution of process and medium. Note that $P_N \in \mathcal{M}_1(\mathcal{M}_1(C[0, T] \times \mathbb{R}))$.

Theorem 1. $(P_N)_{N \geqslant 1}$ satisfies the LDP with rate function

$$I(Q) = H(Q \mid W \otimes \mu) - F(Q) \tag{1.8}$$

where H denotes the relative entropy

$$H(Q \mid W \otimes \mu) = \int dQ \log \frac{dQ}{d(W \otimes \mu)} \tag{1.9}$$

The proof of Theorem 1 will be given in Section 2.1. Roughly, the statement in Theorem 1 means that

$$\frac{1}{N} \log P_N(A) \approx -\inf_{Q \in A} I(Q) \tag{1.10}$$

for large N and for A sufficiently regular. For a precise formulation of the LDP we refer to Deuschel and Stroock,[9] pp. 35–36.

One sees from (1.5) that $F \equiv 0$ when $H_N \equiv 0$ (i.e., $f, g \equiv 0$). Thus $H(Q \mid W \otimes \mu)$ is the rate function for the system without interaction.

1.3. McKean–Vlasov Equation

Before we analyze $I(Q)$, we first give an alternative representation for $F(Q)$ in (1.5) that will turn out to be more convenient. For given $\omega \in \mathbb{R}$ and $q \in \mathcal{M}_1(\mathbb{R} \times \mathbb{R})$ define

$$\beta^{\omega, q}(x) = -\int q(dy, d\pi)\, \hat{f}'(y - x; \omega, \pi) - g'(x; \omega) \qquad (t \in [0, T], x \in \mathbb{R}) \tag{1.11}$$

Let $P^{\omega, Q}$ be the law of the unique (strong) solution of the one-dimensional Itô equation

$$dx_t = \beta^{\omega, \Pi_t Q}(x_t)\, dt + d\xi_t \tag{1.12}$$

where ξ_t is a standard Brownian motion on $\mathbb{R}$ and x_0 has law λ. Here $\Pi_t Q$ is the projection of Q at time t, i.e.,

$$(\Pi_t Q)(E \times F) = Q(\{(x_{[0,T]}, \omega) \colon x_t \in E, \omega \in F\}), \qquad E, F \subset \mathbb{R} \quad (1.13)$$

For fixed ω the drift in (1.12) has a mean-field form, i.e., the interaction in (1.2) of a single-component diffusion with the other components and with the medium appears in (1.12) as an average w.r.t. the given measure $\Pi_t Q$.

Lemma 2. For all Q

$$F(Q) = \int Q(dx_{[0,T]}, d\omega) \log \frac{dP^{\omega,Q}}{dW}(x_{[0,T]}) \tag{1.14}$$

The proof of Lemma 2 will be given in Section 2.2. By combining (1.8), (1.9), and (1.14) we get the following simpler representation for the rate function:

Corollary 1. For all Q

$$I(Q) = H(Q \mid P^Q) \tag{1.15}$$

where $P^Q \in \mathcal{M}_1(C[0, T] \times \mathbb{R})$ is defined by

$$P^Q(dx_{[0,T]}, d\omega) = \mu(d\omega)\, P^{\omega,Q}(dx_{[0,T]}) \tag{1.16}$$

Since $I(Q) \geqslant 0$ for all Q, one sees from (1.10) that as $N \to \infty$ the measure P_N tends to concentrate around the zeros of I, i.e., the solutions of

$$Q = P^Q \tag{1.17}$$

The next theorem states that (1.17) has a unique solution. Define $\nu^Q \in \mathcal{M}_1(\mathbb{R})$ to be the projection of Q on the medium coordinate, i.e.,

$$\nu^Q(F) = Q(\{(x_{[0,T]}, \omega) \colon \omega \in F\}) \qquad (F \in \mathbb{R}) \tag{1.18}$$

Let $Q^\omega \in \mathcal{M}_1(C[0, T])$ be the regular conditional probability measure obtained from Q after conditioning on ω, i.e.,

$$Q(dx_{[0,T]}, d\omega) = \nu^Q(d\omega)\, Q^\omega(dx_{[0,T]}) \tag{1.19}$$

The results that follow will be proved under the following assumption on the initial measure λ for the single-component diffusions[4]:

(A1) λ has a density ϕ w.r.t. Lebesgue measure satisfying $\phi \in L^1(dx) \cap L^p(dx)$ for some $p > 1$.

Theorem 2. Assume (A1). Then (1.17) has a unique solution Q_* which has the following properties:

1. $\nu^{Q_*} = \mu$.
2. Q_*^ω is the law of a Markov diffusion process for μ-a.s. all ω.
3. Let $q_t^\omega = \Pi_t Q_*^\omega$. Then q_t^ω is the weak solution of the McKean–Vlasov equation[5]

$$\begin{cases} \dfrac{\partial}{\partial t} q_t^\omega = \mathscr{L}^\omega q_t^\omega & (t \in (0, T], \omega \in \mathbb{R}) \\ q_0^\omega = \lambda \end{cases} \tag{1.20}$$

where $\mathscr{L}^\omega$ is the nonlinear operator

$$\mathscr{L}^\omega q_t^\omega = -\frac{\partial}{\partial x} [\beta^{\omega, q_t} q_t^\omega] + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_t^\omega \qquad (\omega \in \mathbb{R}) \tag{1.21}$$

and q_t is defined by $q_t(dx, d\omega) = \mu(d\omega)\, q_t^\omega(dx)$.

4. The diffusion process in 2 has generator L_t^ω given by

$$L_t^\omega = \beta^{\omega, q_t} \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \qquad (\omega \in \mathbb{R}) \tag{1.22}$$

The proof of Theorem 2 will be given in Section 2.2. Note that the equations in (1.20) for different values of ω are *coupled*, because

$$\beta^{\omega, q_t}(x) = -\int \mu(d\pi) \int q_t^\pi(dy)\, f'(y - x; \omega, \pi) - g'(x; \omega) \tag{1.23}$$

depends on the whole family $\{q_t^\pi\}_{\pi \in \mathbb{R}}$ [see (1.11)].

[4] Assumption (A1) could in principle be weakened by using the technique of Lyapunov functions, as in Sznitman.[16] However, we stick to (A1) because it allows us to give a rather elementary proof of uniqueness of the solution of (1.17).

[5] Equations (1.20)–(1.21) mean that

$$\frac{d}{dt} \int q_t^\omega(dx)\, \phi(x) = \int q_t^\omega(dx)\, \beta^{\omega, q_t}(x)\, \phi'(x) + \frac{1}{2} \int q_t^\omega(dx)\, \phi''(x)$$

for every $\phi \in \mathscr{D}$, the space of infinitely differentiable functions with compact support. By standard arguments this implies that q_t^ω for $t > 0$ has a density that is a classical solution of (1.20).

As a corollary to Theorems 1 and 2, we obtain the following law of large numbers:

Corollary 2. Assume (A1). Then

$$P_N \Rightarrow \delta_{Q_*} \qquad \text{weakly as} \quad N \to \infty \tag{1.24}$$

1.4. Empirical Flow and Large Deviations

With each $Q \in \mathcal{M}_1(C[0, T] \times \mathbb{R})$ is associated the flow of marginals $q_{[0,T]} = (\Pi_t Q)_{t \in [0,T]}$. Define the *double-layer empirical flow*

$$l_N = \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{(x_t^i, \omega^i)} \right)_{t \in [0,T]} \tag{1.25}$$

This is a random variable taking values in $\mathcal{M}_1(\mathbb{R} \times \mathbb{R})^{[0,T]}$. (The topology on this power set is the one induced by the weak topology on $\mathcal{M}_1(C[0, T] \times \mathbb{R})$ via the map $Q \to q_{[0,T]}$.) Note that both $q_{[0,T]}$ and l_N take values in the subset of $\mathcal{M}_1(\mathbb{R} \times \mathbb{R})^{[0,T]}$ consisting of those flows whose projection on the medium coordinate is independent of t. We shall denote this subset by $\mathcal{M}$. The empirical flow l_N contains less information than the empirical measure L_N [recall (1.3)]. Therefore its large-deviation behavior can be obtained from Theorem 1 via the *contraction principle* (Varadhan,[17] Theorem 2.4).

To formulate the LDP for $(l_N)_{N \geqslant 1}$ we introduce the following notation. For $q_{[0,T]} \in \mathcal{M}$, let $q^{\omega}_{[0,T]}$ be the conditional flow given ω, i.e.,

$$q_t(dx, d\omega) = v^q(d\omega)\, q_t^{\omega}(dx) \qquad (t \in [0, T]) \tag{1.26}$$

where v^q is the projection of q_t on the medium coordinate (which is independent of t). Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, and let $\mathcal{D}^*$ be its dual (the elements of which are distributions). For $\psi^* \in \mathcal{D}^*$ and $p \in \mathcal{M}_1(\mathbb{R})$ define the norm

$$\|\psi^*\|_p^2 = \frac{1}{2} \sup_{\phi \in \mathcal{D}:\, \langle p, \phi'^2 \rangle > 0} \frac{\langle \psi^*, \phi \rangle^2}{\langle p, \phi'^2 \rangle} \tag{1.27}$$

where $\langle \cdot \rangle$ denotes the usual inner product. Let $\Delta \subset \mathcal{M}$ be the set of all flows satisfying

$$\begin{aligned} &v^q \ll \mu \\ &t \to q_t^{\omega} \text{ is weakly differentiable for } v^q\text{-a.s. all } \omega \end{aligned} \tag{1.28}$$

Finally, let

$$\wp_N(\cdot) = \int \mu^{\otimes N}(d\omega)\, P_N^{\omega}(l_N \in \cdot) \tag{1.29}$$

which is the law of l_N under the joint distribution of process and medium. Note that $\wp_N \in \mathcal{M}_1(\mathcal{M})$.

Theorem 3. $(\wp_N)_{N \geqslant 1}$ satisfies the LDP with rate function

$$i(q_{[0,T]}) = \begin{cases} \int_0^T dt \left\{ \int \nu^q(\omega) \left\| \frac{\partial}{\partial t} q_t^{\omega} - \mathscr{L}^{\omega} q_t^{\omega} \right\|_{q_t^{\omega}}^2 \right\} + H(\nu^q \mid \mu) \\ \qquad \text{if} \quad q_{[0,T]} \in \Delta \\ \infty \qquad \text{otherwise} \end{cases} \tag{1.30}$$

The proof of Theorem 3 will be given in Section 2.3. Note that $i(q_{[0,T]}) = 0$ iff $\nu^q = \mu$ and q_t^{ω} is the solution of the McKean–Vlasov equation for μ-a.s. all ω [recall (1.20), (1.21), and (1.23)].

1.5. Central Limit Theorem

It is possible to deduce from Theorem 1 a central limit theorem (CLT) for the empirical measure L_N in (1.3). The general technique is formulated by Bolthausen.[2] Essentially, what we must do is show that the rate function $Q \to I(Q)$ in (1.8) and (1.15) has a strictly positive and finite curvature at its unique zero Q_*. However, in order to apply Bolthausen's theorem we need a technical assumption, namely[6]:

(A2) There are functions α_i, β_i: $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and numbers $c_i \in \mathbb{R}^+$ such that

$$f(y - x; \omega, \pi) = \sum_{i=0}^{\infty} c_i \alpha_i(x, \omega)\, \beta_i(y, \pi) \tag{1.31}$$

with (1) $\sum_i c_i < \infty$; (2) α_i, β_i twice continuously differentiable w.r.t. the variable x, resp. y; and (3) α_i, α_i', α_i'', β_i, β_i', β_i'' bounded uniformly in i.

[6] By applying the techniques in Sznitman,[16] the CLT could in principle be proved without Assumption (A2). However, (i) Bolthausen's method nicely connects large deviations and CLT; (ii) the proof is easily modified to cover other models, e.g., spin-flip systems (see Section 3); (iii) Assumption (A2) is satisfied in many interesting examples [e.g., the Kuramoto model, $f(x; \omega, \pi) = -K \cos x$, $g(x; \omega) = -x\omega$; see also Ben Arous and Brunaud,[1] Section I.1, for a discussion of this assumption and more examples].

Our central limit theorem reads:

Theorem 4. Assume (A2). Let $\mathscr{C}_b$ be the set of bounded continuous functions from $C[0, T] \times \mathbb{R}$ to $\mathbb{R}$. As $N \to \infty$ the field

$$\left(N^{1/2}\left[\int \phi\, dL_N - \int \phi\, dQ_*\right]\right)_{\phi \in \mathscr{C}_b} \tag{1.32}$$

converges under P_N to a Gaussian field with covariance

$$C(\phi, \psi) = \int Q_*(dx_{[0,T]}, d\omega)\, \phi[Q_*](x_{[0,T]}, \omega)\, \psi[Q_*](x_{[0,T]}, \omega) \tag{1.33}$$

where

$$\begin{aligned} &\phi[Q_*](x_{[0,T]}, \omega) \\ &\quad = \phi(x_{[0,T]}, \omega) - \phi^* \\ &\quad\quad - \int_0^T \left(\int Q_*(dy_{[0,T]}, d\pi)\, [\phi(y_{[0,T]}, \pi) - \phi^*]\, \hat{f}'(y_t - x_t; \omega, \pi) \right) dw_t^\omega \end{aligned} \tag{1.34}$$

with $\phi^* = \int \phi\, dQ_*$ (similarly for ψ), $w_t^\omega = x_t - \int_0^t \beta^{\omega, \Pi_s Q_*}\, ds$ (which is a Brownian motion under Q_*^ω), and $\hat{f}$ given by (1.6).

The statement in Theorem 4 means the following: for $\phi_1, \phi_2, \ldots, \phi_n \in \mathscr{C}_b$ the vector

$$\left(N^{1/2}\left[\int \phi_i\, dL_N - \int \phi_i\, dQ_*\right]\right)_{i=1}^n \tag{1.35}$$

converges in law to an n-dimensional Gaussian random variable with mean zero and covariance matrix $(C(\phi_i, \phi_j))_{i,j=1}^n$.

The proof of Theorem 4 will be given in Section 2.4. From the proof it will be seen that the covariance matrix is strictly positive definite.

2. PROOF OF LEMMAS 1 AND 2 AND THEOREMS 1–3

2.1. Proof of Lemma 1 and Theorem 1

Proof of Lemma 1. The proof is based on two basic tools in stochastic calculus, namely Girsanov's formula and Itô's rule (see, e.g., Karatzas and Shreve,[13] Theorems 3.3.3 and 3.5.1). Girsanov's formula yields [recall (1.2)]

$$\frac{dP_N^{\omega}}{dW^{\otimes N}}(\mathbf{x}_{[0,T]}) = \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\int_0^T \left(\frac{\partial H_N}{\partial x^i}(\mathbf{x}_t, \omega)\right)^2 dt \right.$$
$$\left. - \sum_{i=1}^{N}\int_0^T \left(\frac{\partial H_N}{\partial x^i}(\mathbf{x}_t, \omega)\right) d\mathbf{x}_t^i \right] \tag{2.1}$$

Under the measure $W^{\otimes N}$, the process $\mathbf{x}_{[0,T]}$ is N-dimensional Brownian motion (see Section 1.1). Thus, by Itô's rule,

$$\sum_{i=1}^{N}\int_0^T \left(\frac{\partial H_N}{\partial x^i}(\mathbf{x}_t, \omega)\right) d\mathbf{x}_t^i$$
$$= H_N(\mathbf{x}_T, \omega) - H_N(\mathbf{x}_0, \omega) - \frac{1}{2}\sum_{i=1}^{N}\int_0^T \left(\frac{\partial^2 H_N}{\partial (x^i)^2}(\mathbf{x}_t, \omega)\right) dt \tag{2.2}$$

Hence

$$\frac{dP_N^{\omega}}{dW^{\otimes N}}(\mathbf{x}_{[0,T]}) = \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\int_0^T \left\{\left(\frac{\partial H_N}{\partial x^i}(\mathbf{x}_t, \omega)\right)^2 - \frac{\partial^2 H_N}{\partial (x^i)^2}(\mathbf{x}_t, \omega)\right\} dt \right.$$
$$\left. - (H_N(\mathbf{x}_T, \omega) - H_N(\mathbf{x}_0, \omega)) \right] \tag{2.3}$$

The rest of the proof simply consists in inserting the definition of H_N [see (1.1)] and rewriting the resulting expression in terms of the empirical measure L_N [see (1.5)]. This leads to the expression given in (1.4)–(1.6). ∎

Proof of Theorem 1. Let R_N be the law of L_N under the measure $W^{\otimes N} \otimes \mu^{\otimes N}$. Under R_N, the pairs $(x^i_{[0,T]}, \omega^i)$ are i.i.d. random variables. It therefore follows from Sanov's Theorem (Deuschel and Stroock,[9] Theorem 3.2.17) that $(R_N)_{N \geqslant 1}$ satisfies the LDP with rate function $H(Q \mid W \otimes \mu)$ given in (1.9). Now, using Lemma 1, we have [recall (1.4) and (1.7)]

$$P_N(\cdot) = \int \mu^{\otimes N}(d\omega)\, P_N^{\omega}(L_N(d\mathbf{x}_{[0,T]}, \omega) \in \cdot)$$
$$= \int \mu^{\otimes N}(d\omega) \int W^{\otimes N}(d\mathbf{x}_{[0,T]})\, \frac{dP_N^{\omega}}{dW^{\otimes N}}(\mathbf{x}_{[0,T]})$$
$$\times 1\{L_N(d\mathbf{x}_{[0,T]}, \omega) \in \cdot\}$$

$$= \int d(W^{\otimes N} \otimes \mu^{\otimes N}) \exp[NF(L_N)] \, 1\{L_N \in \cdot\}$$

$$= \int R_N(dQ) \exp[NF(Q)] \, 1\{Q \in \cdot\} \tag{2.4}$$

Identity (2.4) means that

$$\frac{dP_N}{dR_N}(Q) = \exp[NF(Q)] \tag{2.5}$$

Our assumption on f, g in Section 1.1 imply that F is a bounded continuous function w.r.t. the weak topology in $\mathcal{M}_1(C[0, T] \times \mathbb{R})$ [see (1.5)]. Therefore, (2.5) allows us to apply Varadhan's Lemma (Varadhan,[17] Theorem 2.2) and conclude that the LDP for $(R_N)_{N \geq 1}$ with rate function $H(Q \mid W \otimes \mu)$ implies the LDP for $(P_N)_{N \geq 1}$ with rate function $H(Q \mid W \otimes \mu) - F(Q)$, as was claimed in (1.8) and (1.9). ∎

2.2. Proof of Lemma 2 and Theorem 2

Proof of Lemma 2. We begin by applying Girsanov's formula to the one-dimensional Itô-equation in (1.12), namely

$$\log \frac{dP^{\omega, Q}}{dW}(x_{[0,T]}) = -\frac{1}{2} \int_0^T (\beta^{\omega, \Pi_t Q}(x_t))^2 \, dt + \int_0^T \beta^{\omega, \Pi_t Q}(x_t) \, dx_t \tag{2.6}$$

We want to show that the r.h.s. of (2.6), when integrated over $Q(d\mathbf{x}_{[0,T]}, d\omega)$, yields $F(Q)$ given in (1.5). Recalling (1.11), we see that the first term in the r.h.s. of (2.6) gives rise to the first term in the r.h.s. of (1.5). To check the remaining terms, let us look a bit closer at the stochastic integral in (2.6).

By (1.11) we have

$$\int Q(dx_{[0,T]}, d\omega) \int_0^T \beta^{\omega, \Pi_t Q}(x_t) \, dx_t$$

$$= -\int Q(dx_{[0,T]}, d\omega)$$

$$\times \int_0^T \left[\int Q(dy_{[0,T]}, d\pi) \, \hat{f}'(y_t - x_t; \omega, \pi) + g'(x_t; \omega) \right] dx_t \tag{2.7}$$

[Note that if $Q \ll W \otimes \mu$, then $x_{[0,T]}$ is a Q-semimartingale, so the stochastic integral in (2.7) makes sense.] Consider the first term in the r.h.s. of (2.7). Since $\hat{f}'$ is an odd function of its first argument, this term equals

$$-\tfrac{1}{2}\int Q(dx_{[0,T]}, d\omega)\int Q(dy_{[0,T]}, d\pi)$$

$$\times\int_0^T \hat{f}'(y_t - x_t; \omega, \pi)\,[dx_t - dy_t] \tag{2.8}$$

We can apply Itô's rule to the two-dimensional semimartingale $(x, y)_{[0,T]}$ and write

$$d\hat{f}(y_t - x_t; \omega, \pi) = \hat{f}''(y_t - x_t; \omega, \pi)\,dt - \hat{f}'(y_t - x_t; \omega, \pi)\,dx_t$$
$$+\hat{f}'(y_t - x_t; \omega, \pi)\,dy_t \tag{2.9}$$

By substituting (2.9) into (2.8) we get the expression

$$-\tfrac{1}{2}\int Q(dx_{[0,T]}, d\omega)\int Q(dy_{[0,T]}, d\pi)$$

$$\times\left[\int_0^T \hat{f}''(y_t - x_t; \omega, \pi)\,dt - \hat{f}(y_T - x_T; \omega, \pi) + \hat{f}(y_0 - x_0; \omega, \pi)\right] \tag{2.10}$$

Next consider the second term in the r.h.s. of (2.7). Itô's rule yields that this term equals

$$-\int Q(dx_{[0,T]}, d\omega)\left[-\tfrac{1}{2}\int_0^T g''(x_t; \omega)\,dt + g(x_T; \omega) - g(x_0; \omega)\right] \tag{2.11}$$

From (2.10) and (2.11) the claim in Lemma 2 easily follows after observing that (1.6) gives

$$\int Q(dx_{[0,T]}, d\omega)\int Q(dy_{[0,T]}, d\pi)\,\hat{f}(y_t - x_t; \omega, \pi)$$

$$= \int Q(dx_{[0,T]}, d\omega)\int Q(dy_{[0,T]}, d\pi)\,f(y_t - x_t; \omega, \pi) \tag{2.12}$$

for every t and, in particular, for $t = 0$ and $t = T$. ∎

Proof of Theorem 2. Observe that $\nu^Q = \nu^{PQ} = \mu$ [recall (1.16)–(1.18)] and that $P^{\omega, Q}$ is the law of the solution of (1.12), i.e., the Markov diffusion with generator given in (1.21). It is therefore easy to see that properties 1–4 in Theorem 2 are satisfied by any solution of (1.17) [note that (1.20) is the Fokker–Planck equation associated with the diffusion Q_*]. Now, the *existence* of a solution of (1.17) comes from the general fact that the rate function of an LDP must have *at least one zero* [Deuschel and Stroock,[9] Exercise 2.1.14(i)]. The *uniqueness* of the solution will be proved in Appendix A. ▌

2.3. Proof of Theorem 3

Let Π denote the map $\Pi: Q \to q_{[0,T]}$ (remember that $q_t = \Pi_t Q$). The topology on $\mathcal{M}$ has been chosen in such a way that Π is continuous. Since $l_N = \Pi L_N$, it follows from the contraction principle (Varadhan,[17] Theorem 2.4) that $(\wp_N)_{N \geq 1}$ satisfies the LDP with rate function

$$j(q_{[0,T]}) = \inf_{\Pi Q = q_{[0,T]}} I(Q) \tag{2.13}$$

We want to show that $j(q_{[0,T]}) = i(q_{[0,T]})$ for every $q_{[0,T]} \in \mathcal{M}$, where i is the rate function given in (1.30). In order to do so, we shall first show that equality holds when $j(q_{[0,T]}) < \infty$ (Steps 1–3 below). After that we shall show that if $i(q_{[0,T]}) < \infty$, then $j(q_{[0,T]}) < \infty$ (Step 4 below), which will complete the proof. The basic ideas are taken from Föllmer[12] (see also Brunaud[4]).

Step 1. By a standard argument involving lower semicontinuity and compactness of the level sets of the rate function I, we have that if $j(q_{[0,T]}) < \infty$, then there exists a Q such that $\Pi Q = q_{[0,T]}$ and $I(Q) = j(q_{[0,T]})$. From (1.8) we have

$$I(Q) = \int v^q(d\omega)\, H(Q^\omega \mid W) + H(v^q \mid \mu) - F(Q) \tag{2.14}$$

Moreover, since $F(Q)$ depends on Q only through $q_{[0,T]}$ [see (1.5) and (1.14)] we have that Q^ω minimizes $H(Q^\omega \mid W)$ under the constraint $\Pi Q^\omega = q^\omega_{[0,T]}$ for v^q-a.s. all ω. As shown by Föllmer,[12] Theorem 1.31, the latter fact implies that Q^ω is the law of a Markov diffusion

$$dx_t = b^\omega_t(x_t)\, dt + dw_t \tag{2.15}$$

for a suitable drift $b^\omega_t(x)$, and that

$$H(Q^\omega \mid W) = \int Q^\omega(dx_{[0,T]}) \int_0^T dt\, [b^\omega_t(x_t)]^2 \tag{2.16}$$

Substituting (2.16) into (2.14), and using Lemma 2 in combination with (2.6) and (2.15), we obtain

$$\begin{aligned} I(Q) &= \tfrac{1}{2}\int \nu^q(d\omega)\int Q^\omega(dx_{[0,T]}) \\ &\quad\times \int_0^T dt\,[b_t^\omega(x_t)-\beta^{\omega,\Pi_tQ}(x_t)]^2 + H(\nu^q\mid\mu) \\ &= \tfrac{1}{2}\int_0^T dt\left\{\int \nu^q(d\omega)\left[\int q_t^\omega(dx)\,(b_t^\omega(x)-\beta^{\omega,\Pi_tQ}(x))^2\right]\right\} \\ &\quad + H(\nu^q\mid\mu) \end{aligned} \tag{2.17}$$

This equation reduces to the required expression in (1.30) if we can show that for every $t\in(0,T]$ and for ν^q-a.s. all ω

$$\frac{1}{2}\int q_t^\omega(dx)\,(b_t^\omega(x)-\beta^{\omega,\Pi_tQ}(x))^2 = \left\|\frac{\partial}{\partial t}q_t^\omega - \mathscr{L}^\omega q_t^\omega\right\|_{q_t^\omega}^2 \tag{2.18}$$

Step 2. To prove (2.18) we proceed as follows. According to (2.15), q_t^ω is the weak solution of the Fokker–Planck equation

$$\frac{\partial q_t^\omega}{\partial t} = -\frac{\partial}{\partial x}[b_t^\omega q_t^\omega] + \frac{1}{2}\frac{\partial^2}{\partial x^2}q_t^\omega \tag{2.19}$$

Together with (1.21) this implies

$$\frac{\partial}{\partial t}q_t^\omega - \mathscr{L}^\omega q_t^\omega = -\frac{\partial}{\partial x}[(b_t^\omega-\beta^{\omega,\Pi_tQ})\,q_t^\omega] \tag{2.20}$$

Hence, recalling the definition of $\|\cdot\|$ in (1.27), we get

$$\begin{aligned} \left\|\frac{\partial}{\partial t}q_t^\omega - \mathscr{L}^\omega q_t^\omega\right\|_{q_t^\omega}^2 &= \frac{1}{2}\sup_{\phi\in\mathscr{D}:\,\langle q_t^\omega,\phi'^2\rangle>0}\frac{\langle(b_t^\omega-\beta^{\omega,\Pi_tQ})\;q_t^\omega,\phi'\rangle^2}{\langle q_t^\omega,\phi'^2\rangle} \\ &\leqslant \frac{1}{2}\langle q_t^\omega,(b_t^\omega-\beta^{\omega,\Pi_tQ})^2\rangle \end{aligned} \tag{2.21}$$

where we have used the Cauchy–Schwarz inequality (recall that $\langle\cdot,\cdot\rangle$ denotes the usual inner product). Thus, to get (2.18) we must show that in (2.21) equality is attained.

Step 3. It suffices to show that the set $\{\phi'\colon\phi\in\mathscr{D}\}$ is dense in $L^2(q_t^\omega)$ for all t and ν^q-a.s. all ω. We first note that q_t^ω is absolutely continuous w.r.t. Lebesgue measure for all t and ν^q-a.s. all ω (this follows from the fact that $Q \ll W \otimes \mu$, $\nu^q \ll \mu$, and the marginals of W are absolutely continuous w.r.t. Lebesgue measure). So, it is enough to prove that if ρ is an absolutely continuous probability measure on $\mathbb{R}$, i.e., $\rho(dx) = p(x)\,dx$, then $\{\phi' : \phi \in \mathscr{D}\}$ is dense in $L^2(\rho)$.

The proof is by contradiction. Suppose $\{\phi' : \phi \in \mathscr{D}\}$ is not dense in $L^2(\rho)$. Then there exists $h \in L^2(\rho)$ such that

$$\int \phi'(x)\, h(x)\, p(x)\, dx = 0 \qquad \text{for every} \quad \phi \in \mathscr{D} \tag{2.22}$$

Since $hp \in L^1(dx)$, it follows from Brezis,[3] Lemma 8.1, that there exists $C \in \mathbb{R}$ such that $hp \equiv C$ a.s. w.r.t. Lebesgue measure. If $C = 0$, then clearly $h \equiv 0$ ρ-a.s. On the other hand, if $C \neq 0$, then $hp \notin L^1(dx)$.

Step 4. To complete the proof of Theorem 3 we need to show that if $i(q_{[0,T]}) < \infty$, then $j(q_{[0,T]}) < \infty$. We use Föllmer,[12] Theorem 1.31, where it is observed that there exists a countable number of bounded continuous functions $(\phi_i)_{i \geqslant 1}$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and a countable (dense) subset $(t_i)_{i \geqslant 1}$ of $[0, T]$ such that $\Pi Q = q_{[0,T]}$ if and only if

$$\int \Pi_{t_i} Q(dx_{[0,T]}, d\omega)\, \phi_i(x, \omega) = 0 \qquad (i = 0, 1, 2,...) \tag{2.23}$$

Now, by compactness and lower semicontinuity of H, for every $n \geqslant 0$ there exists a Q_n such that $H(Q_n \mid W \otimes \mu) < \infty$ and Q_n minimizes $H(Q \mid W \otimes \mu)$ under the constraint that (2.23) holds for $i = 1, 2,..., n$. Since we have proved that $i(q_{[0,T]}) = j(q_{[0,T]})$ when $j(q_{[0,T]}) < \infty$, it follows from (2.13) that

$$I(Q_n) = \inf \left\{ i(p_{[0,T]}) \colon \int p^{\omega}_{t_i}(dx, d\omega)\, \phi_i(x, \omega) = 0 \text{ for } i = 1,..., n \right\} \tag{2.24}$$

In particular, $I(Q_n) \leqslant i(q_{[0,T]})$. By compactness of the level sets of I, the sequence $(Q_n)_{n \geqslant 1}$ has a limit point Q which, by lower semicontinuity of I, satisfies $I(Q) \leqslant i(q_{[0,T]})$. Moreover, (2.23) holds for Q. Hence, via (2.13) we get $j(q_{[0,T]}) \leqslant I(Q) \leqslant i(q_{[0,T]})$. ∎

2.4. Proof of Theorem 4

The proof essentially amounts to applying the method developed by Bolthausen[2] to the random variables

$$X_i = \delta_{(x^i_{[0,T]}, \omega^i)} \qquad (i = 1,..., N) \tag{2.25}$$

Strictly speaking, this method only applies to random variables taking values in certain "nice" Banach spaces, namely Banach spaces of type 2 (such as L^p-spaces with $2 \leqslant p < \infty$). Unfortunately, $\mathcal{M}_1(C[0, T] \times \mathbb{R})$ is not in this class. However, this problem can be circumvented via a trick due to Ben Arous and Brunaud,[1] which consists in mapping $\mathcal{M}_1(C[0, T] \times \mathbb{R})$ into a Banach space of type 2. In this section we *formally* compute the covariance operator according to Bolthausen's recipe (Steps 1–3 below) and check its strict positivity (I–II below), which is the key to having a central limit theorem. The change-of-variable trick, which provides *rigorous justification* for what is done here and which requires the use of Assumption (A2), is given in Appendix B.

Step 1. We start by letting ν_* be the law of the $\mathcal{M}_1(C[0, T] \times \mathbb{R})$-valued random variable $\delta_{x[0,T],\omega)} - Q_*$ induced by Q_*. For $R \in \mathcal{M}_1(C[0, T] \times \mathbb{R})$ and $\phi \in \mathcal{C}_b$ we write $\phi(R) = \int \phi\, dR$ and $\phi^* = \phi(Q_*)$. The *free* covariance operator $(\Gamma(\phi, \psi))_{\phi, \psi \in \mathcal{C}_b}$ is defined by

$$
\begin{aligned}
\Gamma(\phi, \psi) &= \int \phi(R)\, \psi(R)\, \nu_*(dR) \\
&= E^{Q_*}\{[\phi(x_{[0,T]}, \omega) - \phi^*][\psi(x_{[0,T]}, \omega) - \psi^*]\} \\
&= \mathrm{Cov}_{Q_*}(\phi, \psi)
\end{aligned}
\tag{2.26}
$$

The meaning of this operator is that the field

$$
\left(N^{1/2} \left[\int \phi\, dL_N - \phi^* \right] \right)_{\phi \in \mathcal{C}_b}
\tag{2.27}
$$

converges, under $Q_*^{\otimes N}$ as $N \to \infty$, to a Gaussian field with covariance $\Gamma(\phi, \psi)$. This follows from the standard central limit theorem for i.i.d. $\mathbb{R}$-valued random variables.

Step 2. For a given $\phi \in \mathcal{C}_b$, let $\hat{\phi} \in \mathcal{M}_0(C[0, T] \times \mathbb{R})$ be the signed measure on $C[0, T] \times \mathbb{R}$ with zero total mass defined by

$$
\hat{\phi} = \int R\phi(R)\, \nu_*(dR)
\tag{2.28}
$$

i.e., for $A \subset C[0, T] \times \mathbb{R}$ measurable,

$$
\begin{aligned}
\hat{\phi}(A) &= \int R(A)\, \phi(A)\, \nu_*(dR) \\
&= \int Q_*(dx_{[0,T]}, d\omega)\, [\delta_{(x_{[0,T]},\omega)}(A) - Q_*(A)][\phi(x_{[0,T]}, \omega) - \phi^*] \\
&= \mathrm{Cov}_{Q_*}(\mathbf{1}_A, \phi)
\end{aligned}
\tag{2.29}
$$

where $\mathbf{1}_A$ is the characteristic function of A. Then Bolthausen's theorem states that [modulo the change-of-variable trick and some regularity assumptions on the function $Q \to F(Q)$ in (1.5), all to be discussed in Appendix B] the field in (2.27) converges, under P_N as $N \to \infty$, to a Gaussian field with covariance

$$\Delta(\phi, \psi) = \Gamma(\phi, \psi) - D^2 F(Q_*)[\hat{\phi}, \hat{\psi}] \tag{2.30}$$

(recall Lemma 1), *provided* $\Delta(\phi, \phi) > 0$ for all ϕ such that $\hat{\phi} \not\equiv 0$.

Step 3. We remark that

$$\Gamma(\phi, \psi) = D^2 H(Q_* \mid W \otimes \mu)[\hat{\phi}, \hat{\psi}] \tag{2.31}$$

as is easily proved from (1.9) via direct computation [see also (2.34)]. Here the second derivative $D^2 H$ is defined in the usual directional sense (Fréchet derivative). By combining (2.30) and (2.31) with (1.8), we get

$$\Delta(\phi, \psi) = D^2 I(Q_*)[\hat{\phi}, \hat{\psi}] \tag{2.32}$$

Thus the requirement $\Delta(\phi, \phi) > 0$ can be interpreted as saying that the rate function $Q \to I(Q)$ must have strictly positive finite curvature at its unique minimum Q_*.

The rest of the proof consists in showing the following two facts. Let $C(\phi, \psi)$ be the covariance defined in (1.33). Then

$$\begin{aligned} &\text{I.} \quad C(\phi, \psi) = \Delta(\phi, \psi) \\ &\text{II.} \quad C(\phi, \phi) > 0 \text{ for all } \phi \text{ such that } \hat{\phi} \not\equiv 0 \end{aligned} \tag{2.33}$$

Proof of I. For simplicity we assume $\phi = \psi$. The proof for the general case follows the same argument.

We first note that, by (2.29), $\hat{\phi} \ll Q_*$ and

$$\frac{d\hat{\phi}}{dQ_*} = \phi - \phi^* \tag{2.34}$$

Using the expression [recall (1.14) and (2.6)]

$$F(Q) = E^Q \left\{ -\tfrac{1}{2} \int_0^T (\beta^{\omega, \Pi, Q}(x_t))^2 \, dt + \int_0^T \beta^{\omega, \Pi, Q}(x_t) \, dx_t \right\} \tag{2.35}$$

we get, by a lengthy but straightforward computation via (1.11),

$$D^2F(Q)[\hat{\phi}, \hat{\phi}] = -E^Q \int_0^T [\gamma^{\omega,\Pi_t\hat{\phi}}(x_t)]^2\, dt$$

$$-2\int \hat{\phi}(dx_{[0,T]}, d\omega) \int_0^T \beta^{\omega,\Pi_t Q}(x_t)\, \gamma^{\omega,\Pi_t\hat{\phi}}(x_t)\, dt$$

$$+2\int \hat{\phi}(dx_{[0,T]}, d\omega) \int_0^T \gamma^{\omega,\Pi_t\hat{\phi}}(x_t)\, dx_t \tag{2.36}$$

with

$$\gamma^{\omega,\Pi_t\hat{\phi}}(x) = \int \hat{\phi}(dy_{[0,T]}, d\pi)\, \hat{f}'(y_t - x; \omega, \pi) \tag{2.37}$$

[The computation becomes elementary once we realize that, due to (2.34), the Itô integrals make sense under $\hat{\phi}$.]

Now let

$$w_t^\omega = x_t - \int_0^t \beta^{\omega,\Pi_s Q_*}\, ds$$

(which is a Brownian motion under Q_*^ω). Then by (2.26), (2.30), (2.34), and (2.36) we have

$$\begin{aligned}
\Delta(\phi, \phi) &= \Gamma(\phi, \phi) - D^2F(Q_*)[\hat{\phi}, \hat{\phi}] \\
&= E^{Q_*}\{[\phi(x_{[0,T]}, \omega) - \phi^*]^2\} + E^{Q_*}\left\{\int_0^T [\gamma^{\omega,\Pi_t\hat{\phi}}(x_t)]^2\, dt\right\} \\
&\quad + 2E^{Q_*}\left\{[\phi(x_{[0,T]}, \omega) - \phi^*] \int_0^T \gamma^{\omega,\Pi_t\hat{\phi}}(x_t)\, dw_t^\omega\right\} \\
&= E^{Q_*}\{[\phi(x_{[0,T]}, \omega) - \phi^*]^2\} + E^{Q_*}\left\{\left[\int_0^T \gamma^{\omega,\Pi_t\hat{\phi}}(x_t)\, dw_t^\omega\right]^2\right\} \\
&\quad + 2E^{Q_*}\left\{[\phi(x_{[0,T]}, \omega) - \phi^*] \int_0^T \gamma^{\omega,\Pi_t\hat{\phi}}(x_t)\, dw_t^\omega\right\} \\
&= E^{Q_*}\left\{\left[\phi(x_{[0,T]}, \omega) - \phi^* + \int_0^T \gamma^{\omega,\Pi_t\hat{\phi}}(x_t)\, dw_t^\omega\right]^2\right\} \\
&= C(\phi, \phi)
\end{aligned} \tag{2.38}$$

where in the second equality we have used the standard isometry property of integration w.r.t. Brownian motion.[7] ∎

Proof of II. Suppose $\phi \in \mathscr{C}_b$ is such that $C(\phi, \phi) = 0$. It is not restrictive to assume $\phi^* = 0$. We want to show that $\hat{\phi} \equiv 0$, i.e., $\phi = 0$ Q_*-a.s. Define the following σ-field on $C[0, T] \times \mathbb{R}$:

$$\mathscr{F}_t = \sigma\{x_s: 0 \leqslant s \leqslant t\} \otimes \mathscr{B} \tag{2.39}$$

with $\mathscr{B}$ denoting the Borel σ-field on $\mathbb{R}$. Let

$$\phi_t(x_{[0,t]}, \omega) = E^{Q_*}\{\phi \mid \mathscr{F}_t\} \tag{2.40}$$

According to (1.33)–(1.34), $C(\phi, \phi) = 0$ implies

$$\phi(x_{[0,T]}, \omega) = \int_0^T \left[\int Q_*(dy_{[0,T]}, d\pi)\, \phi(y_{[0,T]}, \pi) \times \hat{f}'(y_t - x_t; \omega, \pi) \right] dw_t^\omega \quad Q_*\text{-a.s.} \tag{2.41}$$

Taking conditional expectation and using the fact that the integral in the r.h.s. of (2.41) is an $\mathscr{F}_t$-martingale, we get

$$\phi_t(x_{[0,t]}, \omega) = \int_0^t \left[\int Q_*(dy_{[0,T]}, d\pi)\, \phi_t(y_{[0,t]}, \pi) \times \hat{f}'(y_s - x_s; \omega, \pi) \right] dw_s^\omega \quad Q_*\text{-a.s.} \tag{2.42}$$

Thus, using again the isometry property of integration w.r.t. Brownian motion, we obtain

$$\begin{aligned} \|\phi_t\|^2_{L^2(Q_*)} &= \left\| \int_0^t \left[\int Q_*(dy_{[0,T]}, d\pi)\, \phi_t(y_{[0,t]}, \pi) \hat{f}'(y_s - x_s; \omega, \pi) \right] dw_s^\omega \right\|^2_{L^2(Q_*)} \\ &= E^{Q_*} \left\{ \int_0^t \left[\int Q_*(dy_{[0,T]}, d\pi)\, \phi_t(y_{[0,t]}, \pi) \hat{f}'(y_s - x_s; \omega, \pi) \right]^2 dt \right\} \\ &\leqslant t\, \|\hat{f}'\|^2_\infty\, \|\phi_t\|^2_{L^2(Q_*)} \end{aligned} \tag{2.43}$$

[7] Let $(w_t)_{t \in [0,T]}$ be a Brownian motion. Let $(\xi_t)_{t \in [0,T]}$ be a stochastic process, adapted to the filtration generated by $(w_t)_{t \in [0,T]}$, such that $E(\int_0^T \xi_t^2\, dt) < \infty$. Then the following equality holds: $E(\int_0^T \xi_t^2\, dt) = E([\int_0^T \xi_t\, dw_t]^2)$.

which implies $\phi_t = 0$ Q_*-a.s. for $t \in [0, 1/\|\hat{f}'\|_\infty^2)$. It is easy to see that this argument can be repeated, and so we get $\phi_t = 0$ Q_*-a.s. for $t \in [0, T]$. Since $\phi_T = \phi$ the conclusion follows. ∎

3. SPIN-FLIP SYSTEMS

All the results stated in Section 1, together with their proofs in Section 2, can be modified in an essentially straightforward manner to cover the case of spin-flip systems. In this section we formulate these modifications and indicate which parts of their proofs are not trivially obtained from the corresponding parts for diffusions. We follow the same order as in Section 1.

3.1. The Model

Let H_N: $\{-1, +1\}^N \times \mathbb{R}^N \to \mathbb{R}$ be the N-particle Hamiltonian given by

$$H_N(\mathbf{x}, \boldsymbol{\omega}) = \frac{1}{2N} \sum_{i,j=1}^{N} f(\omega^i, \omega^j)\, x^i x^j + \sum_{i=1}^{N} g(\omega^i)\, x^i \tag{3.1}$$

where $\mathbf{x} = (x^i)_{i=1}^N$ is the state variable and $\boldsymbol{\omega} = (\omega^i)_{i=1}^N$ is the medium variable. As for diffusions, the ω^i are i.i.d. random variables with common law μ. Moreover, the functions f, g are assumed to be bounded and continuous.

For given $\boldsymbol{\omega}$, let $\mathbf{x}_t = (x_t^i)_{i=1}^N$ be the N-spin system defined to be the Markov chain with infinitesimal generator $\mathscr{G}$, acting on functions ϕ: $\{-1, +1\}^N \to \mathbb{R}$ as follows:

$$(\mathscr{G}\phi)(\mathbf{x}) = \sum_{i=1}^{N} c_N^{\boldsymbol{\omega}}(i, \mathbf{x})[\phi(\mathbf{x}^i) - \phi(\mathbf{x})] \tag{3.2}$$

Here, $\mathbf{x}^i$ is the state obtained from $\mathbf{x}$ by flipping the ith spin x^i, and

$$\begin{aligned} c_N^{\boldsymbol{\omega}}(i, \mathbf{x}) &= \exp\left[\frac{1}{2}\{H_N(\mathbf{x}, \boldsymbol{\omega}) - H_N(\mathbf{x}^i, \boldsymbol{\omega})\}\right] \\ &= \exp\left[\frac{1}{N} \sum_{j=1, j\neq i}^{N} \hat{f}(\omega^i, \omega^j)\, x^i x^j + g(\omega^i)\, x^i\right] \end{aligned} \tag{3.3}$$

with $\hat{f}(\omega, \pi) = f(\omega, \pi) + f(\pi, \omega)$. For every $\boldsymbol{\omega}$, (3.2) has a reversible equilibrium measure proportional to $\exp[-H_N(\mathbf{x}, \boldsymbol{\omega})]$. The initial condition $\mathbf{x}_0$ is assumed to have product distribution $\lambda^{\otimes N}$, where λ is any probability measure on $\{-1, +1\}$. The path space for a single spin is $D[0, T]$, the space of right-continuous piecewise-constant functions from $[0, T]$ to $\{-1, +1\}$. This space has a topology and a Borel σ-field, provided by the Skorohod metric; see, e.g., Ethier and Kurtz,[10] p. 117.

We denote by $W^{\otimes N}$ the law of the N-spin system whose generator has the form (3.2) with $c_N^{\boldsymbol{\omega}} \equiv 1$. All other notations introduced in Section 1 ($P_N^{\boldsymbol{\omega}}, L_N, P_N, \ldots$) are left unchanged.

3.2. Empirical Measure and Large Deviations

The analogs of Lemma 1 and Theorem 1 read as follows.

Lemma 3. For given $\boldsymbol{\omega}$

$$\frac{dP_N^{\boldsymbol{\omega}}}{dW^{\otimes N}}(\mathbf{x}_{[0,T]}) = \exp[NF(L_N(\mathbf{x}_{[0,T]}, \boldsymbol{\omega})) + O(1)] \tag{3.4}$$

where for $Q \in \mathcal{M}_1(D[0, T] \times \mathbb{R})$

$$\begin{aligned} F(Q) = &\int Q(dx_{[0,T]}, d\omega) \\ &\times \left\{ \int_0^T dt \left(1 - \exp\left[\int Q(dy_{[0,T]}, d\pi)\, \hat{f}(\omega, \pi)\, x_t y_t + g(\omega)\, x_t\right]\right)\right. \\ &\left. + \tfrac{1}{2} \int Q(dy_{[0,T]}, d\pi)\, [\hat{f}(\omega, \pi)(x_T y_T - x_0 y_0) + g(\omega)(x_T - x_0)] \right\} \end{aligned} \tag{3.5}$$

The proof of Lemma 3 relies on Girsanov's formula for spin-flip systems, which is easily derived from Girsanov's formula for point processes (see Comets[5] or Lipster and Shiryaev,[15] Theorem 19.3).

Theorem 5. $(P_N)_{N \geqslant 1}$ satisfies the LDP with rate function

$$I(Q) = H(Q \mid W \otimes \mu) - F(Q) \tag{3.6}$$

This follows from Lemma 3 as for diffusions. The technical difference is that the martingale term in the Girsanov formula is not driven by a Brownian motion, but by a compensated Poisson process.

3.3. McKean–Vlasov Equation

Given $Q \in \mathcal{M}_1(D[0, T] \times \mathbb{R})$ and $\omega \in \mathbb{R}$, let $P^{\omega, Q}$ be the law of the single-spin system whose initial distribution is λ and whose rate of flipping from x to $-x$ at time t is given by $c^{\omega,\Pi_t Q}(x)$, where for $q \in \mathscr{M}_1(\{-1,1\} \times \mathbb{R})$

$$c^{\omega,q}(x) = \exp\left[x\left(\int q(dy, d\pi)\, f(\omega, \pi)\, y + g(\omega)\right)\right] \tag{3.7}$$

In analogy with Lemma 2 and Corollary 1, the next facts are easily proved.

Lemma 4. For all Q

$$F(Q) = \int Q(dx_{[0,T]}, d\omega) \log \frac{dP^{\omega,Q}}{dW}(x_{[0,T]}) \tag{3.8}$$

Corollary 3. For all Q

$$I(Q) = H(Q \mid P^Q) \tag{3.9}$$

where $P^Q \in \mathscr{M}_1(D[0,T] \times \mathbb{R})$ is defined by

$$P^Q(dx_{[0,T]}, d\omega) = \mu(d\omega)\, P^{\omega,Q}(dx_{[0,T]}) \tag{3.10}$$

The next theorem is the analog of Theorem 2. Define ν^Q as in (1.18).

Theorem 6. Equation (3.9) has a unique solution Q_* which has the following properties:

1. $\nu^{Q_*} = \mu$.
2. Q_*^ω is the law of a Markov chain on $\{-1, +1\}$ for μ-a.s. all ω.
3. Let $q_t^\omega = \Pi_t Q_*^\omega$. Then q_t^ω solves the differential equation

$$\begin{cases} \dfrac{\partial}{\partial t} q_t^\omega = \mathscr{L}^\omega q_t^\omega & (t \in (0, T], \omega \in \mathbb{R}) \\ q_0^\omega = \lambda \end{cases} \tag{3.11}$$

where L^ω is the nonlinear operator

$$(\mathscr{L}^\omega q_t^\omega)(x) = q_t^\omega(-x)\, c^{\omega,q_t}(-x) - q_t^\omega(x)\, c^{\omega,q_t}(x) \quad (\omega \in \mathbb{R}) \tag{3.12}$$

and q_t is defined by $q_t(x, d\omega) = \mu(d\omega)\, q_t^\omega(x)$.

4. Under Q_*^ω the rate of flipping from x to $-x$ at time t for the Markov chain in 2 is c^{ω,q_t}.

The only essential difference from the proof of Theorem 2 is the part concerning the uniqueness of the solution of (3.11), which is much easier here. Indeed, via the relation $q_t^\omega(-1) + q_t^\omega(+1) = 1$ for all ω and t, (3.11) can be rewritten as an equation for $q_t^\omega(+1)$, thought of as an element of $L^\infty(\mu)$. The *coupled* family of equations in (3.11), indexed by $\omega \in \mathbb{R}$, is an ordinary differential equation in the Banach space $L^\infty(\mu)$ driven by a locally Lipschitz vector field. Uniqueness follows by classical arguments (Brezis,[3] Theorem VII.3).

3.4. Empirical Flow and Large Deviations

The definitions of l_N and $\wp_N$ are the same as in Section 1 [see (1.25) and (1.26)]. For p a probability measure on $\{-1, +1\} \times \mathbb{R}$ and $\omega \in \mathbb{R}$, define Ψ_p^ω: $\mathbb{R}^{\{-1,+1\}} \to \mathbb{R}^+$ by

$$\Psi_p^\omega(\lambda) = \sup_{\delta \in \mathbb{R}^{\{-1,+1\}}} \left\{ \sum_{x=\pm 1} \left[\lambda(x)\, \delta(x) - p^\omega(x)\, c^{\omega,p}(x)(e^{\hat{\delta}(x)} - \hat{\delta}(x) - 1) \right] \right\} \tag{3.13}$$

where $\hat{\delta}(x) = \delta(-x) - \delta(x)$. Defining Δ as in (1.28), we obtain the following analog of Theorem 3.

Theorem 7. $(\wp_N)_{N \geqslant 1}$ satisfies the LDP with rate function

$$i(q_{[0,T]}) = \begin{cases} \displaystyle\int_0^T dt \left\{ \int \nu^q(d\omega)\, \Psi_{q_t}^\omega \left(\frac{\partial q_t^\omega}{\partial t} - \mathscr{L}^\omega q_t^\omega \right) \right\} + H(\nu^q \mid \mu) \\ \qquad \text{if} \quad q_{[0,T]} \in \Delta \\ \infty \qquad \text{otherwise} \end{cases} \tag{3.14}$$

For the model *without* random field a different representation for i is given by Comets.[5]

The proof of Theorem 7 is not a trivial modification of the proof of Theorem 3. We therefore give a sketch here (Steps 1–3 below).

Step 1. Fix a flow $q_{[0,T]} \in \Delta$. Suppose that there exists a $Q \in \mathscr{M}_1(D[0, T] \times \mathbb{R})$ such that $I(Q) < \infty$ and Q minimizes I under the constraint $\Pi_t Q = q_t$ for $t \in [0, T]$. Then, as for diffusions, it can be shown that Q^ω is Markovian for μ almost all ω (e.g., by using the notion of h-process; see Föllmer,[12] Theorem 1.31). Let us denote by $k_t^\omega(x_t)$ the flip rate of this process at time t. Then from Girsanov's formula for spin processes we get

$$I(Q) = \int_0^T dt \left\{ \int \nu^q(d\omega) \right.$$
$$\left. \times \left[\sum_{x=\pm 1} q_t^\omega(x) \left(c^{\omega,q_t}(x) - k_t^\omega(x) + k_t^\omega(x) \log \frac{k_t^\omega(x)}{c^{\omega,q_t}(x)} \right) \right] \right\} \tag{3.15}$$

Step 2. Write the identity

$$\sum_{x=\pm 1} q_t^\omega(x)\left(c^{\omega,q_t}(x)-k_t^\omega(x)+k_t^\omega(x)\log\frac{k_t^\omega(x)}{c^{\omega,q_t}(x)}\right)$$
$$=\sup_{\delta\in\mathbb{R}^{\{-1,+1\}}}\sum_{x=\pm 1} q_t^\omega(x)[\delta(x)(k_t^\omega(x)-c^{\omega,q_t}(x))$$
$$-c^{\omega,q_t}(x)(e^{\delta(x)}-\delta(x)-1)] \tag{3.16}$$

which is easily checked by noting that the supremum is attained at $\delta=\delta_*$ given by $\delta_*(x)=\log[k_t^\omega(x)/c^{\omega,q_t}(x)]$. We claim that the r.h.s. of (3.16) equals

$$\sup_{\delta\in\mathbb{R}^{\{-1,+1\}}}\sum_{x=\pm 1} q_t^\omega(x)[\hat\delta(x)(k_t^\omega(x)-c^{\omega,q_t}(x))-c^{\omega,q_t}(x)(e^{\hat\delta(x)}-\hat\delta(x)-1)] \tag{3.17}$$

[which is the same as the r.h.s. of (3.16), but with δ replaced by $\hat\delta$]. This will be shown below. From (3.17), together with the identities

$$\sum_{x=\pm 1} q_t^\omega(x)\,\hat\delta(x)[k_t^\omega(x)-c^{\omega,q_t}(x)]$$
$$=\sum_{x=\pm 1}\delta(x)\{q_t^\omega(x)[k_t^\omega(x)-c^{\omega,q_t}(x)]\}$$
$$=\sum_{x=\pm 1}\delta(x)\left[\frac{\partial}{\partial t}q_t^\omega(x)-\mathscr{L}^\omega q_t^\omega(x)\right] \tag{3.18}$$

we get $I(Q)=i(q_{[0,T]})$. The second equality in (3.18) uses (3.11) and (3.12) with k_t^ω replacing c^{ω,q_t}. The proof can now be completed as for Theorem 3.

Step 3. We still have to show that (3.16) equals (3.17), which amounts to verifying that $\delta_*=\hat\gamma$ for some $\gamma\in\mathbb{R}^{\{-1,+1\}}$. This is equivalent to saying that $\sum_{x=\pm 1}\delta_*(x)=0$ or

$$k_t^\omega(x)=c^{\omega,q_t}(x)\,e^{\lambda_t x}\qquad\text{for some}\quad \lambda_t\in\mathbb{R} \tag{3.19}$$

There are various ways of checking (3.19). The most direct and elementary way consists in looking for the minimum of (3.15) [w.r.t. the rates $k_t^\omega(x)$] under the constraint

$$\frac{\partial q_t^\omega(x)}{\partial t}=q_t^\omega(-x)\,k_t^\omega(-x)-q_t^\omega(x)\,k_t^\omega(x),\qquad t\in(0,T] \tag{3.20}$$

The classical method of Lagrange multipliers shows that the k_t^ω realizing the minimum must have the form (3.19) (we already know that the minimum exists). The details are straightforward.

Theorem 7 shows that the large deviations for the empirical flow are controlled by the positive convex functions Ψ_p^ω. These are *not* norms squared, unlike the case for diffusions. To appreciate the analogy between Theorem 3 and Theorem 7, note that we could have used in Theorem 3 the following expression equivalent to (1.27)[8]:

$$\|\psi^*\|_p^2 = \sup_{\phi \in \mathscr{D}} \{\langle \psi^*, \phi \rangle - \tfrac{1}{2}\langle p, \phi'^2 \rangle\} \tag{3.21}$$

3.5. Central Limit Theorem

The CLT for spin systems will be proved under the following assumption, which, for technical reasons that will be explained in Appendix B, is much stronger than the corresponding Assumption (A2) for diffusions:

(A3) There exist a finite set $X \subset \mathbb{R}$ and functions α_i, β_i: $\mathbb{R} \to X$ $(i = 1,..., p)$ such that

$$f(\omega, \pi) = \sum_{i=1}^{p} \alpha_i(\omega)\, \beta_i(\pi) \tag{3.22}$$

We note that Assumption (A3) is satisfied in two relevant cases: (i) when f is constant, i.e., the medium does not affect the interaction [e.g., the Curie–Weiss model with $f(\omega, \pi) = -\beta$, $g(\omega) = -\omega$]; (ii) when the support of the medium law μ is finite.

For $x_{[0,T]} \in D[0, T]$, we let $J_t(x_{[0,T]})$ be the number of jumps of the path $x_{[0,T]}$ up to and including time t.

Theorem 8. Let $\mathscr{C}_b$ be the set of bounded continuous functions from $D[0, T] \times \mathbb{R}$ to $\mathbb{R}$. As $N \to \infty$ the field

$$\left(N^{1/2}\left[\int \phi\, dL_N - \int \phi\, dQ_*\right]\right)_{\phi \in \mathscr{C}_b} \tag{3.23}$$

converges under P_N to a Gaussian field with covariance

$$C(\phi, \psi) = \int Q_*(dx_{[0,T]}, d\omega)\, \phi[Q_*](x_{[0,T]}, \omega)\, \psi[Q_*](x_{[0,T]}, \omega) \tag{3.24}$$

where

$$\begin{aligned}
&\phi[Q_*](x_{[0,T]}, \omega) \\
&\quad = \phi(x_{[0,T]}, \omega) - \phi^* \\
&\quad\quad + \int_0^T \left(\int Q_*(dy_{[0,T]}, dx)\, [\phi(y_{[0,T]}, \pi) - \phi^*]\, y_t \hat{f}(\omega, \pi)\right) dw_t^\omega
\end{aligned} \tag{3.25}$$

with $\phi^* = \int \phi \, dQ_*$ (similarly for ψ) and

$$w_t^\omega = J_t(x_{[0,T]}) - \int_0^t c^{\omega, \Pi_s Q_*}(x_s)\, ds$$

(which is a martingale under Q_*^ω).

The part of the proof of the CLT for diffusions, contained in Section 2.4, extends readily to spin systems. The part concerning the change-of-variable trick is sketched at the end of Appendix B.

APPENDIX A

We prove here that Eq. (1.17) has a unique solution. We assume (A1): the initial measure λ has a density ϕ w.r.t. Lebesgue measure satisfying $\phi \in L^1(dx) \cap L^p(dx)$ for some $p > 1$.

Step 1. A Priori Estimate. We first prove that if Q_* is a solution of (1.17), then there are constants $A > 0$ and $0 \leqslant \alpha < 1/2$ such that

$$q_t^\omega(x) \leqslant \frac{A}{t^\alpha} \qquad \text{for every} \quad x, \omega \in \mathbb{R} \text{ and } t > 0 \tag{A.1}$$

where $q_t^\omega = \Pi_t Q_*^\omega$. To see this, observe that $Q_* = P^{Q_*}$ gives

$$\frac{dQ_*^\omega}{dW} = \frac{dP^{\omega, Q_*}}{dW} \tag{A.2}$$

The process having law P^{ω, Q_*} is a diffusion whose drift $\beta_t^{\omega, \Pi_t Q_*}$ is the bounded derivative of a bounded function [recall (1.11)–(1.12)]. By the usual argument involving Girsanov's formula and Itô's rule, one sees that there is a constant $B > 0$ such that the Radon–Nikodym derivative in (A.2) is bounded by B uniformly in ω. It follows that

$$q_t^\omega(x) \leqslant B\psi_t(x) \qquad (t > 0) \tag{A.3}$$

where $\psi_t = \Pi_t W$, i.e.,

$$\psi_t(x) = \frac{1}{\sqrt{2\pi t}} \int e^{-(1/2t)(x-y)^2} \phi(y)\, dy \tag{A.4}$$

By Hölder's inequality we have

$$\psi_t(x) \leqslant \frac{1}{\sqrt{2\pi t}} \left[\int e^{-(q/2t)(x-y)^2}\, dy \right]^{1/q} \|\phi\|_p = \frac{C}{t^{1/2 - 1/2q}} \|\phi\|_p \tag{A.5}$$

with $C>0$ some constant and $1/p+1/q=1$. Now (A.1) follows from (A.3) and (A.5).

Step 2. Uniqueness. Let Q and $\tilde{Q}$ be two solutions of (17), with q_t^ω and $\tilde{q}_t^\omega$ denoting the corresponding marginals. As mentioned in footnote 5, these are both classical solutions of the McKean–Vlasov equation (1.20). Define, for $t>0$,

$$F_t^\omega(x)=q_t^\omega(x)-\tilde{q}_t^\omega(x) \tag{A.6}$$

The following relation is easily checked [see (1.11) and (1.20)–(1.21)]:

$$\frac{\partial F_t^\omega}{\partial t}-\frac{1}{2}\frac{\partial^2 F_t^\omega}{\partial x^2}=\frac{\partial L_t^\omega}{\partial x} \tag{A.7}$$

where

$$L_t^\omega(x)=F_t^\omega(x)\int dy\,\mu(d\pi)\,q_t^\pi(y)(\hat{f}'(y-x;\omega,\pi)+g'(x;\omega))$$
$$+\tilde{q}_t^\omega(x)\int dy\,\mu(d\pi)\,F_t^\pi(x)(\hat{f}'(y-x;\omega,\pi)+g'(x;\omega)) \tag{A.8}$$

Now let $G(x,t)$ be the fundamental solution of the heat equation, i.e.,

$$G(x,t)=\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t} \tag{A.9}$$

Then (A.7) yields

$$\begin{aligned}F_t^\omega(x)&=\int_0^t ds\int dy\,G(x-y,t-s)\frac{\partial}{\partial y}L_s^\omega(y)\\&=-\int_0^t ds\int dy\,L_s^\omega(y)\frac{\partial G}{\partial y}(x-y,t-s)\end{aligned} \tag{A.10}$$

where the last integration by parts is justified since, for $\omega\in\mathbb{R}$ and $t>0$, $L_t^\omega(x)$ is a bounded function of x. Now define

$$H_t^\omega=\int|F_t^\omega(x)|\,dx \tag{A.11}$$

By substituting (A.1) into (A.8), one obtains the following estimate:

$$\int|L_t^\omega(x)|\,dx\leqslant\frac{A}{t^\alpha}H_t^\omega+\frac{B}{t^\alpha}\int\mu(d\pi)\,H_t^\mu\qquad\text{for all }\omega\text{ and }t \tag{A.12}$$

with A and B suitable constants independent of t. Moreover, by direct computation one sees that there is a constant K such that

$$\int \left| \frac{\partial G}{\partial y}(x-y, t-s) \right| dx \leqslant \frac{K}{\sqrt{t-s}} \qquad \text{for all } y \tag{A.13}$$

Putting together (A.10)–(A.13) and defining $H_t = \int \mu(d\omega)\, H_t^\omega$, we get

$$H_t \leqslant C \int_0^t \frac{1}{s^\alpha \sqrt{t-s}} H_s\, ds \qquad (t \in [0, T]) \tag{A.14}$$

with C some constant independent of t. Below we shall show that (A.14) implies $H_t \equiv 0$. We complete the proof by showing how the latter implies $Q = \tilde{Q}$. Indeed, if $H_t \equiv 0$, then (A.6) and (A.11) give $q_t^\omega(x) = \tilde{q}_t^\omega(x)$ for all t and for almost every ω, x. With Q and $\tilde{Q}$ being solutions of (1.17), this in turn implies that for almost every $\omega \in \mathbb{R}$ the diffusions with laws Q^ω and $\tilde{Q}^\omega$ have the same bounded and continuous drift and the same initial distribution. By standard uniqueness results for stochastic differential equations, it follows that $Q^\omega = \tilde{Q}^\omega$ ω-a.s., and so $Q = \tilde{Q}$.

Step 3. $H_t \equiv 0$. Let us define

$$\|H\|_t = \sup_{s \in [0,t]} H_s \tag{A.15}$$

By (A.14)

$$H_s \leqslant C \|H\|_t \int_0^t \frac{ds}{s^\alpha \sqrt{t-s}} \qquad \text{for all} \quad s \in [0, t] \tag{A.16}$$

Now, because $\alpha < 1/2$ we have

$$\lim_{t \to 0} \int_0^t \frac{ds}{s^\alpha \sqrt{t-s}} = 0 \tag{A.17}$$

This, together with (A.16), implies that there exists $t' > 0$ such that $H_t = 0$ for $t \in [0, t']$. Using (A.14) again, we obtain

$$H_s \leqslant \frac{C}{t'^\alpha} \int_{t'}^t \frac{ds}{\sqrt{t-s}} \tag{A.18}$$

It is trivial to see that

$$\lim_{t \to t'} \int_{t'}^t \frac{ds}{\sqrt{t-s}} = 0 \tag{A.19}$$

and so there must exist $t''>0$ such that $H_t=0$ also for $t\in[t', t'+t'']$. This argument can be repeated to show that $H_t=0$ for $t\in[t'+t'', t'+2t'']$ and so on. Hence $H_t\equiv 0$.

We remark that $\alpha<1/2$ in (A.1) is a consequence of our assumption (A1) on the initial condition λ. By removing that assumption we would get $\alpha=1/2$ and the proof would not work.

APPENDIX B

The proof of Theorem 4 will be completed here, i.e., we carry out the change-of-variable trick which provides the rigorous justification for the formal computation in Section 2.4. We first give an outline of the proof, which is based on Claims 1–4 below. The proof of these claims comes later. At the end of this Appendix we show what modifications are needed for spin-flip systems.

Let $\mathcal{M}(C[0, T]\times\mathbb{R})$ be the vector space of signed measures on $C[0, T]\times\mathbb{R}$, provided with the weak topology.

Claim 1. There exists a Banach space $(B, \|\cdot\|)$, a continuous linear map $T\colon \mathcal{M}(C[0, T]\times\mathbb{R})\to B$, and a continuous map $\Psi\colon B\to\mathbb{R}$ that is bounded on $T(\mathcal{M}_1(C[0, T]\times\mathbb{R}))$ and infinitely Fréchet differentiable, such that

$$\frac{dP_N^{\omega}}{dW^{\otimes N}}(\mathbf{x}_{[0,T]})=\exp[N\Psi(T(L_N))] \tag{B.1}$$

Moreover, $\mathrm{Range}(T^*)\subset\mathscr{C}_b$, where $T^*\colon B^*\to(\mathcal{M}(C[0, T]\times\mathbb{R}))^*$ is the adjoint map of T.

Next, let

$$Y_i=T(\delta_{(x^i_{[0,T]},\omega^i)})\qquad (i=1,..., N) \tag{B.2}$$

and denote by p_N and w_N the laws of $\mathbf{Y}=(Y_1,..., Y_N)$ induced by P_N, resp. $W^{\otimes N}\otimes\mu^{\otimes N}$. Then it follows from (B.1) that

$$\frac{dp_N}{dw_N}(\mathbf{Y})=\exp[N\Psi(M_N)] \tag{B.3}$$

with $M_N=N^{-1}\sum_{i=1}^N Y_i$.

As we shall see later, the Banach space $(B, \|\cdot\|)$ in Claim 1 satisfies the requirements of Bolthausen's theorem [see (B.11) below], which can therefore be applied to the random variables Y_i with the help of (B.3).

Moreover, by the Contraction Principle, the p_N-law of M_N satisfies the LDP with rate function $J(Y) = \inf_{T(Q)=Y} I(Q)$, which has a unique zero at $Y_* = T(Q_*)$.

To compute the covariance of the corresponding CLT, we begin by defining a probability measure p on B by putting

$$\frac{dp}{dw}(Y) = \frac{1}{Z}\exp[D\Psi(Y_*)[Y]] \tag{B.4}$$

where w is the law of $T(\delta_{(x[0,T],\omega)})$ induced by $W\otimes\mu$, D is the Fréchet derivative, and Z is the normalizing constant.

Claim 2. The measure p is the law of $T(\delta_{(x[0,T],\omega)})$ induced by Q_*.

Next, let $p_* = p - Y_*$. For $h, k \in B^*$ define

$$\begin{aligned} \gamma(h,k) &= \int p_*(dY)\, h(Y)\, k(Y) \\ \tilde{h} &= \int p_*(dY)\; Yh(Y) \in B \end{aligned} \tag{B.5}$$

Claim 3. Let Γ be as in (2.31). The following identities hold:

$$\begin{aligned} \gamma(h,k) = \Gamma(T^*h, T^*k) &= D^2H(Q_*)[\widehat{T^*h}, \widehat{T^*h}] \\ \tilde{h} &= T(\widehat{T^*h}) \\ D^2\Psi(Y_*)[\tilde{h}, \tilde{k}] &= D^2F(Q_*)[\widehat{T^*h}, \widehat{T^*k}] \end{aligned} \tag{B.6}$$

Thus, by what was shown in Section 2.4 (proof of *II*), we have $\gamma(h,h) - D^2\Psi(Y_*)[\tilde{h}, \tilde{h}] > 0$ unless $\tilde{h} \equiv 0$. It follows from Bolthausen's theorem that, under the p_N-law as $N \to \infty$, the field

$$(N^{1/2}h(M_N - Y_*))_{h\in B^*} = \left(N^{1/2}\int (T^*h)\, d(L_N - Q_*)\right)_{h\in B^*} \tag{B.7}$$

converges weakly to a Gaussian field with covariance [recall (2.38)]

$$\gamma(h,k) - D^2\Psi(Y_*)[\tilde{h}, \tilde{k}] = D^2I(Q_*)[\widehat{T^*h}, \widehat{T^*h}] = C(T^*h, T^*k) \tag{B.8}$$

To complete the proof of Theorem 4 it therefore suffices to show the following fact.

Claim 4. For given $\phi_1,\dots,\phi_n \in \mathscr{C}_b$, $n \in \mathbb{N}$, the Banach space $(B, \|\cdot\|)$ and the map T can be constructed in such a way that $\{\phi_1,\dots,\phi_n\} \subset \text{Range}(T^*)$.

We next proceed with the proof of Claims 1–4.

Proof of Claims 1 and 4. By redefining the functions α_i, β_i in Assumption (A2), it is clear that instead of (1.31) we may also write

$$-\hat{f}(y-x;\omega,\pi)-g(x;\omega)=\sum_{i=0}^{\infty} c_i\alpha_i(x,\omega)\,\beta_i(y,\pi) \tag{B.9}$$

[where $(\alpha_i, \beta_i, c_i)_{i\geqslant 0}$ have the properties described in Assumption (A2)]. Substituting (B.9) into (1.5), we get

$$\begin{aligned}
F(Q)= -\tfrac{1}{2}\int_0^T dt \Bigg[&\sum_{i,j} c_i c_j \left(\int Q(dx_{[0,T]}, d\omega)\,\alpha_i'(x_t,\omega)\,\alpha_j'(x_t,\omega)\right)\\
&\times\left(\int Q(dy_{[0,T]}, d\pi)\,\beta_i(y_t,\pi)\right)\left(\int Q(dy_{[0,T]}, d\pi)\,\beta_j(y_t,\pi)\right)\\
&+\sum_i c_i \left(\int Q(dx_{[0,T]}, d\omega)\,\alpha_i''(x_t,\omega)\right)\\
&\times\left(\int Q(dy_{[0,T]}, d\pi)\,\beta_i(y_t,\pi)\right)\Bigg]\\
&+\tfrac{1}{2}\sum_i c_i \left(\int Q(dx_{[0,T]}, d\omega)\,\alpha_i(x_T,\omega)\right)\\
&\times\left(\int Q(dy_{[0,T]}, d\pi)\,\beta_i(y_T,\pi)\right)\\
&-\tfrac{1}{2}\sum_i c_i \left(\int Q(dx_{[0,T]}, d\omega)\,\alpha_i(x_0,\omega)\right)\\
&\times\left(\int Q(dy_{[0,T]}, d\pi)\,\beta_i(y_0,\pi)\right)
\end{aligned} \tag{B.10}$$

Next, denote by c the finite measure on $\mathbb{N}$ given by $c(\{i\})=c_i$. We introduce the following Banach spaces:

$$\begin{aligned}
B_1 &= L^3(\mathbb{N}^2\times[0,T], c^{\otimes 2}\otimes dt)\\
B_2 &= L^2(\mathbb{N}\times[0,T], c\otimes dt)\\
B_3 &= L^2(\mathbb{N}\cup\{-1,-2,\dots,-n\}, c+\delta_{-1}+\cdots+\delta_{-n})\\
B &= (B_1)^3\times(B_2)^2\times(B_3)^4
\end{aligned} \tag{B.11}$$

The norm $\|\cdot\|$ on B will be chosen to be supremum of the norms on the factors. An element $Y \in B$ will be written

$$Y = (Y_1^1, Y_1^2, Y_1^3, Y_2^1, Y_2^2, Y_3^1, Y_3^2, Y_3^3, Y_3^4) \tag{B.12}$$

The map $T\colon \mathscr{M}(C[0, T] \times \mathbb{R}) \to B$ is now defined as follows: For $i, j \in \mathbb{N}$ and $t \in [0, T]$

$$\begin{aligned}
T(Q)_1^1(i, j, t) &= \int Q(dx_{[0,T]}, d\omega)\, \alpha_i'(x_t, \omega)\, \alpha_j'(x_t, \omega) \\
T(Q)_1^2(i, j, t) &= \int Q(dx_{[0,T]}, d\omega)\, \beta_i(x_t, \omega) \\
T(Q)_1^3(i, j, t) &= \int Q(dx_{[0,T]}, d\omega)\, \beta_j(x_t, \omega) \\
T(Q)_2^1(i, t) &= \int Q(dx_{[0,T]}, d\omega)\, \alpha_i''(x_t, \omega) \\
T(Q)_2^2(i, t) &= \int Q(dx_{[0,T]}, d\omega)\, \beta_i(x_t, \omega)
\end{aligned} \tag{B.13}$$

For $i \in \mathbb{N}$,

$$\begin{aligned}
T(Q)_3^1(i) &= \int Q(dx_{[0,T]}, d\omega)\, \alpha_i(x_T, \omega) \\
T(Q)_3^2(i) &= \int Q(dx_{[0,T]}, d\omega)\, \beta_i(x_T, \omega) \\
T(Q)_3^2(i) &= \int Q(dx_{[0,T]}, d\omega)\, \alpha_i(x_0, \omega) \\
T(Q)_3^4(i) &= \int Q(dx_{[0,T]}, d\omega)\, \beta_i(x_0, \omega)
\end{aligned} \tag{B.14}$$

For $i = 1, 2, \ldots, n$ and $k = 1, 2, 3, 4$

$$T(Q)_3^k(-i) = \int Q(dx_{[0,T]}, d\omega)\, \phi_i(x_{[0,T]}, \omega) \tag{B.15}$$

A straightforward computation (which we omit) allows us to get an explicit (but rather long) formula for the operator

$$T^*\colon B^* = (B_1^*)^3 \times (B_2^*)^2 \times (B_3^*)^4 \to (\mathcal{M}(C[0, T] \times \mathbb{R}))^*$$

from which it easily follows that $\mathrm{Range}(T^*) \subset \mathscr{C}_b$. Moreover, we see from (B.15) that

$$\phi_i = T^*(0, 0, 0, 0, 0, 0, 0, 0, \mathbf{1}_{\{-i\}}) \tag{B.16}$$

which proves Claim 4.

For $Y \in B$ define

$$\begin{aligned}\Psi(Y) = &-\tfrac{1}{2} \int_{\mathbb{N}^2 \times [0, T]} (dc^{\otimes 2} \otimes dt)\; Y_1^1 Y_1^2 Y_1^3 \\ &-\tfrac{1}{2} \int_{\mathbb{N} \times [0, T]} (dc \otimes dt)\; Y_2^1 Y_2^2 \\ &+\tfrac{1}{2} \int_{\mathbb{N}} dc\, (Y_3^1 Y_3^2 - Y_3^3 Y_3^4)\end{aligned} \tag{B.17}$$

Clearly, Ψ is continuous and infinitely Fréchet differentiable. Moreover, Ψ is bounded on $T(\mathcal{M}_1(C[0, T] \times \mathbb{R}))$ because the components of $T(Q)$ are bounded uniformly in $Q \in \mathcal{M}_1(C[0, T] \times \mathbb{R})$. Finally, (B.10) and (B.13)–(B.15) imply that $F(Q) = \Psi(T(Q))$ (note that F extends to all $\mathcal{M}(C[0, T] \times \mathbb{R})$). This proves Claim 1.[8] ▮

Proof of Claim 2. The main step in the proof is the relation

$$DF(Q_*)[\delta_{(x_{[0,T]},\omega)}] = \log \frac{dQ_*}{d(W \otimes \mu)} (x_{[0,T]}, \omega) \qquad \text{for} \quad W \otimes \mu\text{-a.s. all } (x_{[0,T]}, \omega) \tag{B.18}$$

This relation is easily obtained from (1.5) by direct computation using the Girsanov formula. We omit the details.

By (B.18) and the fact that T is linear and continuous, we have

$$D\Psi(Y_*)[T(\delta_{(x_{[0,T]},\omega)})] = DF(Q_*)[\delta_{(x_{[0,T]},\omega)}] \tag{B.19}$$

[8] As we mentioned earlier, Bolthausen's theorem can be used with no further assumption in Banach spaces of type 2 (see Ben Arous and Brunaud[1] for the precise definition). Now, L^p-spaces with $2 \leqslant p < \infty$ are of type 2, and finite products of Banach spaces of type 2 are again of type 2. Thus our $(B, \|\cdot\|)$ defined in (B.11) is a Banach space of type 2.

Thus, for any ρ: $B \to \mathbb{R}$ measurable and bounded, (B.4) gives

$$\begin{aligned}
&\int p(dY)\,\rho(Y)\\
&\quad = \frac{1}{Z}\int w(dY)\,\rho(Y)\exp\{D\Psi(Y_*)[Y]\}\\
&\quad = \frac{1}{Z}\int (W\otimes\mu)\,(dx_{[0,T]}, d\omega)\,\rho(T(\delta_{(x_{[0,T]},\omega)}))\,\frac{dQ_*}{d(W\otimes\mu)}\,(x_{[0,T]},\omega)\\
&\quad = \frac{1}{Z}\int Q_*(dx_{[0,T]}, d\omega)\,\rho(T(\delta_{(x_{[0,T]},\omega)})) \qquad\qquad \text{(B.20)}
\end{aligned}$$

Letting $\rho \equiv 1$, we get $Z = 1$. ∎

Proof of Claim 3. Using Claim 2 and the definition of adjoint operator, we have

$$\begin{aligned}
\gamma(h,k) &= \int p(dY)\,h(Y-Y_*)\,k(Y-Y_*)\\
&= \int Q_*(dx_{[0,T]}, d\omega)\,h(T(\delta_{(x_{[0,T]},\omega)} - Q_*))\,k(T(\delta_{(x_{[0,T]},\omega)} - Q_*))\\
&= \int Q_*(dx_{[0,T]}, d\omega)\,[T^*h(x_{[0,T]},\omega) - E^{Q_*}(T^*h)]\\
&\qquad \times [T^*k(x_{[0,T]},\omega) - E^{Q_*}(T^*k)]\\
&= \Gamma(T^*h, T^*k) \qquad\qquad \text{(B.21)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{h} &= \int p(dY)\,(Y-Y_*)\,h(Y-Y^*)\\
&= \int Q_*(dx_{[0,T]}, d\omega)\,T(\delta_{(x_{[0,T]},\omega)} - Q_*)\,h(T(\delta_{(x_{[0,T]},\omega)} - Q_*))\\
&= T\left(\int Q_*(dx_{[0,T]}, d\omega)\,(\delta_{(x_{[0,T]},\omega)} - Q_*)\,T^*h(\delta_{(x_{[0,T]},\omega)} - Q_*)\right)\\
&= T(\widehat{T^*h}) \qquad\qquad \text{(B.22)}
\end{aligned}$$

where we again use the notation $(T^*h)(Q)$ for $\int (T^*h)\,dQ$. The third identity in (B.6) follows from the second and the fact that T is linear and continuous. ∎

We finally sketch the corresponding change-of-variable trick for spin-flip systems. We only show the key part of the construction, which consists in defining a linear continuous map T from $\mathcal{M}(D[0, T] \times \mathbb{R})$ to a Banach space $(B, \|\cdot\|)$ of type 2 and a smooth function $\Psi: B \to \mathbb{R}$ such that $F = \Psi \circ T$. The rest of the proof is a simple modification of what we have done above for diffusions.

In order to avoid unnecessary complications, we shall explain the construction for the function F' defined by

$$F'(Q) = \int Q(dx_{[0,T]}, d\omega) \times \int_0^T dt \exp\left[\int Q(dy_{[0,T]}, d\pi)\, f(\omega, \pi)\, x_t\, y_t\right] \tag{B.23}$$

The extension of our construction from F' to F [defined in (3.5)] is straightforward.

In the above argument for diffusions, we were able to map $\mathcal{M}(C[0, T] \times \mathbb{R})$ to a Banach space $(B, \|\cdot\|)$ that is a *finite* product of L^p-spaces with $p \geqslant 2$ and therefore is a Banach space of type 2. In doing so, we used the fact that the function $F(Q)$ in (1.5) is "polynomial" in Q [i.e., $F(\lambda Q)$, $\lambda \in \mathbb{R}$, is a polynomial in λ]. Such a property holds neither for F in (3.5) nor for F' in (B.23). Here is where Assumption (A3) plays a crucial role. Since the function $\{-1, +1\} \times \mathbb{R} \to \mathbb{R}$ given by $(x, \omega) \mapsto x\alpha_i(\omega)$ assumes only finite values, we can find a $q \in \mathbb{N}$ and smooth functions ϕ_j^i, ψ_j^i, $j = 1,..., q$, such that for all $z \in \mathbb{R}$

$$e^{\alpha_i(\omega) xz} = \sum_{j=1}^{q} \psi_j^i(x\alpha_i(\omega))\, \phi_j^i(z) \qquad (i = 1,..., p) \tag{B.24}$$

Substituting (3.22) into (B.23) and using (B.24), we find

$$\begin{aligned} F'(Q) &= \int_0^T dt \int Q(dx_{[0,T]}, d\omega) \\ &\quad \times \prod_{i=1}^{p} \prod_{j=1}^{q} \psi_j^i(x_t \alpha_i(\omega))\, \phi_j^i\left(\int Q(dy_{[0,T]}, d\pi)\, y_t \beta_i(\pi)\right) \\ &= \sum_{j_1,..., j_p = 1}^{q} \int_0^T dt \left\{\left[\prod_{i=1}^{p} \int Q(dx_{[0,T]}, d\omega)\, \psi_{j_i}^i(x_t \alpha_i(\omega))\right]\right. \\ &\quad \left. \times \left[\prod_{i=1}^{p} \phi_{j_i}^i\left(\int Q(dy_{[0,T]}, d\pi)\, y_t \beta_i(\pi)\right)\right]\right\} \end{aligned} \tag{B.25}$$

Note that the arguments of the functions ϕ_j^i in (B.25) are bounded uniformly in Q. Thus it is not restrictive to assume these functions and all their derivatives to be bounded. We now define

$$B = (L^{p+1})^{q^p} \otimes (L^{p+1})^p \tag{B.26}$$

The norm $\|\cdot\|$ on B is taken to be the supremum of the norms on the factors. An element $f \in B$ is written in the form

$$f = ((f_{\mathrm{j}}^{(1)}), (f_i^{(2)}))_{\mathrm{j} \in \{1,\ldots,q\}^{\{1,\ldots,p\}},\, i \in \{1,\ldots,p\}} \tag{B.27}$$

The maps T: $\mathscr{M}(D[0, T] \times \mathbb{R}) \to B$ and Ψ: $B \to \mathbb{R}$ are defined by

$$\begin{aligned} T(Q)_{\mathrm{j}}^{(1)}(t) &= \int Q(dx_{[0,T]}, d\omega) \\ &\quad \times \prod_{i=1}^{p} \psi_{j_i}^i(x_t \alpha_i(\omega)) \qquad (\mathrm{j} \in \{1,\ldots, q\}^{\{1,\ldots,p\}}) \\ T(Q)_i^{(2)}(t) &= \int Q(dy_{[0,T]}, d\pi)\, y_t \beta_i(\pi) \qquad (i \in \{1,\ldots, p\}) \\ \Psi(f) &= \sum_{\mathrm{j}} \int_0^T dt \left[f_{\mathrm{j}}^{(1)}(t) \prod_{i=1}^{p} \phi_{j_i}^i(f_i^{(2)}(t)) \right] \end{aligned} \tag{B.28}$$

It is easily seen that T is linear and continuous. Moreover, the smoothness of Ψ follows from the fact that the functions ϕ_j^i and their derivatives are Lipschitz continuous. Finally, it is clear that $F' = \Psi \circ T$ and that B, being a finite product of L^p-spaces, is of type 2.

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